

# UNIFORMIZATION OF NONLINEAR HAMILTONIAN SYSTEMS OF VLASOV AND HARTREE TYPE

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**ABSTRACT.** Nonlinear Hamiltonian systems describing the abstract Vlasov and Hartree equations are considered in the framework of algebraic Poissonian theory. The concept of uniformization is introduced; it generalizes the method of second quantization of classical systems to arbitrary Hamiltonian (Lie-Jordan, or Poissonian) algebras, in particular to the algebras of operators in indefinite spaces. A functional calculus is developed for the uniformized observables, which unifies the calculi of generating functionals for classical, Bosonian and Fermionian multiparticle variables. The nonlinear kinetic equation of Vlasov and Hartree type are derived for both classical and quantum multiparticle Hamiltonian systems in the mean field approximation. The possibilities for finding approximate solutions of these equations from the solutions of uniformized linear equations are investigated.

## 1. INTRODUCTION

Recently, much interest [1] has been devoted to studying the region of the spectrum of the Hamiltonians of certain self-interacting boson fields corresponding to particle-like (soliton) solutions of the classical mean field non-linear equations. Such excitations can be naturally interpreted as composite stable quantum particles that have finite energy even in the classical limit. As a comparison [2] of the results for exactly solvable models showed, there is a direct correspondence between the quantum [3] and classical [4] particle-like solutions. Frequently, it is simplest to obtain such solutions directly for the quantum case, and not for the corresponding classical case, for which they were obtained only very recently by the solution of the inverse scattering problem.

This correspondence can be viewed as a heuristic basis of an asymptotic method for finding soliton solutions of nonlinear equations in not only classical field theory but also for the difference equations that describe the dynamics of nonlinear crystals, and also certain nonlinear equations of quantum electrodynamics. This method, which we call the uniformization method, can be regarded mathematically as a new method of linearization leading to

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asymptotically exact solutions of nonlinear equations in the large number limit by the infinite increase in the number of degrees of freedom.

Applied to ordinary nonlinear Schrödinger equations, the uniformization method essentially reduces to second quantization and the construction of a quasiclassical soliton solution from the solution of the  $n$ -particle equation for sufficiently large  $n$ . However, this interpretation no longer holds for the spaces with indefinite metric. Applying this method to the nonlinear equation of quantum electrodynamics ([5], p.11) in which the local time enters as a fourth coordinate, we arrive at a many-time formalism that reduces the solution of these nonlinear equations to the solution of ultrahyperbolic equations that may not have independent physical meaning.

Therefore, we set forth the uniformization method on an abstract algebraic basis without referring to its physical interpretation which is contained in the concrete functional realizations. As the object of uniformization, we choose the abstract Vlasov equation, which generalizes the classical and quantized Vlasov equations considered in [5], and also equations that reduce to equations of Hartree type in spaces with indefinite metric.

For the classical case, the uniformization method is essentially the transition from a field system described by the Vlasov kinetic equation to a system with variable number of particles, i.e., it is the procedure which is the inverse of the derivation of the Vlasov equation from the Bogolyubov chain of  $n$ -particle equations [6]. In the quantum case, the uniformization method gives a unified algebraic basis for the method of second quantization of bosons and fermions with a superselection rule that avoids the consideration of unphysical - odd - observables that do not commute with the particle number operator. The developed calculus of representing functionals for such even operators combines and replaces the calculus of symbols of commuting variables (Bargmann [6]) and anticommuting variables (Berezin [7]). In such an approach, the difference between bosons and fermions does not reside in the operators but only in the multiparticle tensor-states.

In the general case it is shown that the algebra of polynomial observables of the uniformized system is isomorphic to the direct sum of the tensor powers of the Hamiltonian algebra generated by the elementary observables of the uniformized system. In the case of algebras of operators in a space with indefinite metric, such a representation generalizes the Fock to a pseudo-Fock representation.

First, we set forth the basic concepts and the definitions relating to nonlinear Hamiltonian systems of Vlasov and Hartree type, and also the uniformization method, showing that such nonlinear kinetic equations naturally appear as characteristic equations in the mean-field approximation of the uniformized Hamiltonian infinite systems. Then we consider the possibility of finding approximate solutions of these nonlinear equations, in particular approximate soliton solutions from the corresponding solutions of the uniformized equations.

## 2. ABSTRACT VLASOV AND HARTREE EQUATIONS

2.1. We consider a dynamical system whose states are described by the elements  $\rho$  of a real linear space  $\mathcal{L}$  as the predual space of linear functionals  $A \mapsto \langle \rho, A \rangle$  on an abstract Hamiltonian algebra  $\mathcal{A}$  with respect to some bilinear form  $\langle \rho, A \rangle$  having real values for all  $\rho \in \mathcal{L}$  and  $A = A^* \in \mathcal{A}^1$ . By the Hamiltonian algebra we mean any complex-linear unital  $\star$ -algebra  $\mathcal{A}$  in which besides the “ordinary”, not necessarily commutative and associative, product  $A \cdot A^* \in \mathcal{A}$ , there is also defined the Poisson brackets  $\{H, A\}$  as a bilinear antisymmetric map  $\mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$  with the property of continuous derivation

$$\{H, A \cdot A^*\} = A \cdot \{H, A^*\} + \{H, A\} \cdot A^*$$

having anti-Hermitian values  $\{A^*, A\}^* = -\{A^*, A\}$  in contrast to  $(A \cdot A^*)^* = A \cdot A^*$  (for more details see [7]). If the ordinary product is associative, the Hamiltonian algebra is said to be Poissonian. In the commutative case  $A \cdot A^* = A^* \cdot A$  the real part  $\Re \mathcal{A} = \{A = A^*\}$  of the Hamiltonian algebra is a real Hamiltonian subalgebra such that  $\mathcal{A}$  can be considered as a complexification of  $\mathcal{R} = \Re \mathcal{A}$  which is usually taken to be a real Lie-Jordan algebra [8]. If the algebra  $\mathcal{A}$  is both commutative and associative with respect to the product  $A \cdot B$  like the algebra of smooth scalar-valued functions on a Hamiltonian phase space, it is called classical Hamiltonian, or, equivalently Poissonian algebra.

We give the name Vlasov Hamiltonian to every real element  $H = H^* \in \mathcal{A}$  that may depend continuously on  $\rho \in \mathcal{L}$  as the derivative  $\delta\gamma(\rho) = H(\rho)$  of some, in general nonlinear  $\mathcal{A}$ -smooth functional  $\gamma : \rho \mapsto \gamma(\rho)$ , called Hamiltonian functional on  $\mathcal{L}$  ( $\gamma$  has real number values  $\gamma(\rho) \in \mathbb{R}$ ). By  $\mathcal{A}$ -smooth we mean such smooth functional which has the derivative as a continuous functional  $\rho \mapsto \delta\gamma(\rho)$  defined for each  $\rho \in \mathcal{L}$  as an element  $H(\rho)$  of  $\mathcal{A}$  such that

$$(2.1) \quad \langle \sigma, \delta \rangle \gamma(\rho) := \frac{\partial \gamma(\rho + t\sigma)}{\partial t} |_{t=0} = \langle \sigma, H(\rho) \rangle, \quad \forall \sigma \in \mathcal{L}.$$

Let us assume that the linear operator  $\rho \mapsto \{\rho, H\}$  is well-defined for each value  $H = H(\rho)$  of a Vlasov Hamiltonian  $H$  as the predual to the derivation  $A \mapsto \{H, A\}$  on the algebra  $\mathcal{A}$  such that<sup>2</sup>

$$(2.2) \quad \langle \{\rho, H\}, A \rangle = \langle \rho, \{H, A\} \rangle, \quad \forall \rho \in \mathcal{L}, A \in \mathcal{A}.$$

We give the name “abstract Vlasov equation” to the evolution equation in the state space  $\mathcal{L} \ni \rho(t)$  of the form

$$(2.3) \quad \frac{\partial \rho(t)}{\partial t} = \{\rho(t), H(t, \rho(t))\}.$$

<sup>1</sup>One may assume that  $\mathcal{L}$  is a Fréchet space with the complexified dual of all linear continuous functionals  $\rho \mapsto \langle A, \rho \rangle$  given by a  $\star$ -algebra  $\mathcal{A}$ .

<sup>2</sup>To this end one may assume that the derivation  $A \mapsto \{H, A\}$ , as well as the action  $A \mapsto H \cdot A$  are  $\sigma(\mathcal{A}, \mathcal{L})$ -continuous for each  $H \in \mathcal{A}$ .

where  $H(t, \rho)$  is the derivative  $\delta\gamma(t, \rho)$  of a nonlinear time-dependent  $\mathcal{A}$ -smooth functional  $\gamma(t, \rho)$  which may, in general, depend on time  $t$ .

**Proposition 1.** *Equation (2.3) is a classical Hamiltonian system with respect to the Hamiltonian functional  $\gamma(t, \rho)$  which has at least one linear integral of motion  $\nu(\rho) = \langle \rho, I \rangle$  generated by the identity  $I \in \mathcal{A}$ .*

Indeed, for any differentiable functional  $\alpha : \rho \mapsto \alpha(\rho)$  that does not depend explicitly on  $t$  we have in accordance with (2.3)

$$(2.4) \quad \frac{\partial \alpha(\rho(t))}{\partial t} = \langle \{\rho(t), H(t, \rho(t))\}, \delta\alpha(\rho(t)) \rangle.$$

Using the definitions (2.2) of the dual operator  $A \mapsto \{H, A\}$  and remembering that  $H = \delta\gamma(\rho)$ , we can write (2.4) in the form  $\frac{\partial \alpha}{\partial t} = \{\gamma, \alpha\}_{cl}$ , where

$$(2.5) \quad \{\gamma, \alpha\}_{cl}(\rho) = \langle \rho, \{\delta\gamma(\rho), \delta\alpha(\rho)\} \rangle$$

is a classical Poisson bracket with respect to the pointwise product  $\alpha\beta$  since

$$\{\gamma, \alpha\beta\}_{cl}(\rho) = \alpha(\rho) \{\gamma, \beta\}_{cl}(\rho) + \{\gamma, \alpha\}_{cl}(\rho)\beta(\rho).$$

This means that the algebra  $\mathfrak{A}_{cl}$  of smooth functionals  $\alpha : \mathcal{L} \rightarrow \mathbb{C}$  is a classical Hamiltonian algebra with commutative and associative Jordan product  $(\alpha \cdot \beta)(\rho) = \alpha(\rho)\beta(\rho)$  and Lie product (2.5) which generates on the manifold  $\mathcal{L}$  a symplectic structure with respect to which (2.3) is simply the Hamiltonian equation with the Hamilton functional  $\gamma(t, \rho)$ . Substituting  $\nu(\rho) = \langle \rho, I \rangle$  as  $\alpha(\rho)$  in (2.5) and remembering that in accordance with the derivation property we have  $\{H, I\} = 0$  for any  $H \in \mathcal{A}$ , we find that  $\nu(\rho)$  is an integral for (2.3):  $\{\gamma, \nu\}_{cl} = 0$ .

Note that the elements  $\alpha$  of the algebra  $\mathfrak{A}_{cl}$  can be interpreted as nice (smooth) observables of the described classical system, taking real or complex values  $\alpha(\rho)$  in the states  $\rho \in \mathcal{L}$ , which should be regarded as deterministic point-states corresponding to zero variances  $|\alpha|^2(\rho) - |\alpha(\rho)|^2 = 0$  simultaneously for all  $\alpha \in \mathfrak{A}_{cl}$ . The Hamiltonian algebra  $\mathcal{A}$ , whose elements generate “elementary classical observables” - the linear functional observables  $\alpha_A = \langle \rho, A \rangle$  - is here not assumed to be classical.

2.2. Actually to the Vlasov equation itself there corresponds only the case when  $\mathcal{A}$  is the classical Poissonian algebra of smooth complex functions  $A(\mathbf{q}, \mathbf{p})$  on the single-particle phase space  $\mathbb{R}^{2d} \ni (\mathbf{q}, \mathbf{p})$  with ordinary pointwise multiplication and involution given by complex conjugation  $A^*(\mathbf{q}, \mathbf{p}) = \overline{A(\mathbf{q}, \mathbf{p})}$ , and the classical Poisson brackets  $\{A^*, A\} = i[A^*, A]_{cl}$ , where

$$[A^*, A]_{cl}(\mathbf{q}, \mathbf{p}) = i(\partial_{\mathbf{q}} A^* \cdot \partial_{\mathbf{p}} A - \partial_{\mathbf{q}} A \cdot \partial_{\mathbf{p}} A^*)(\mathbf{q}, \mathbf{p})$$

is the classical commutator, i.e. the Poisson bracket divided by  $i = \sqrt{-1}$ . The space  $\mathcal{L}$  of real density functions  $\rho(\mathbf{q}, \mathbf{p})$  absolutely integrable in the product with any  $A \in \mathcal{A}$  is dual to  $\mathcal{A}$  with respect to the phase space integral

$$\langle \rho, A \rangle = \int \int \rho(\mathbf{q}, \mathbf{p}) A(\mathbf{q}, \mathbf{p}) d\mathbf{q} d\mathbf{p}.$$

Thus the classical Vlasov equation (2.3) can be regarded as the Liouville equation with Hamilton function  $H(\mathbf{p}, \mathbf{q}, t, \rho)$  that in general continuously depends on the density function  $\rho$  (actually in the original Vlasov equation  $H(\mathbf{p}, \mathbf{q}, t, \rho)$  is an affine function of  $\rho$  corresponding to a quadratic form of the functional  $\rho \mapsto \gamma(t, \rho)$ ).

One can obtain a different class of the classical Hamiltonian systems of the form (2.3) by considering the quantized Vlasov equations as in [5], or even a more general equations when  $\mathcal{A}$  is an operator algebra in a complex-linear Hilbert or pre-Hilbert space  $\mathcal{H}$  with scalar product  $(\cdot|\cdot)$ , taking  $\mathcal{L}$  as the space of density operators  $\rho$  dual to  $\mathcal{A}$  with respect to the trace-pairing  $\langle \rho, A \rangle = \text{Tr}(\rho A)$ . The Poisson bracket with respect to the noncommutative operator product in  $\mathcal{A}$  can be defined by the usual commutator  $[A^*, A] = A^*A - AA^*$  as  $\{A^*, A\} = i[A^*, A]$ , so that (2.3) takes the form of a quantized Liouville, or von Neumann equation

$$\frac{\partial \rho(t)}{\partial t} = i[\rho(t), H(t, \rho(t))],$$

with  $\rho$ -dependent Hamilton operator  $H(t, \rho)$ . Note that the real part  $\Re \mathcal{A}$  of the noncommutative operator algebra  $\mathcal{A}$  is not invariant under the associative operator product  $AB$ , but it is a nonassociative real Hamiltonian algebra with respect to the symmetrized operator product  $A \cdot B = \frac{1}{2}(AB + BA)$ .

One can also have intermediate semiquantized cases obtained, for example, by partial quantization of the Vlasov equation. In all these cases, if the states  $\rho$  satisfy the positivity condition  $\langle \rho, A^*A \rangle \geq 0$  and the normalization condition  $\langle \rho, I \rangle = 1$  they can have a probability interpretation by regarding the forms  $\langle \rho, A \rangle$  not as exact values of elementary observables but as expected values of the results of individual measurements of these observables. However this probabilistic interpretation cannot be extended onto the nonlinear kinetic equation (2.3) even though they preserve the positivity of  $\rho$  and its normalization, since they do not preserve statistical mixtures except in the linear case. As we shall see, the the nonlinear kinetic equations describe not the individual but a collective dynamics of the infinite ensemble of the corresponding classical or quantum particles in the mean field approximation.

Finally, our treatment also encompasses the (general) case when  $\mathcal{A}$  is an operator  $\star$ -algebra in a space  $\mathcal{H}$  with indefinite metric, i.e., when  $(\cdot|\cdot)$  is an arbitrary nondegenerate Hermitian form (indefinite scalar product). In this case, the set of operators of the form  $AA^*$  does not form a cone, and the states  $\rho$  cannot have a probability interpretation of the individual particles. They can only be regarded as pure states of the corresponding classical field system, or as a quasi-classical limit of an infinite quantum field system. Numerous examples of such a situation arise in a quasiclassical treatment of boson and fermion fields with indefinite commutation relations in [9].

Suppose  $\psi \mapsto \psi^*$  is a canonical mapping of the space  $\mathcal{H}$  into the dual space  $\mathcal{H}^*$  defined by the equation  $\psi^*\varphi = (\varphi|\psi)$ ,  $\forall \varphi \in \mathcal{H}$ , and  $\psi\psi^*$  are one-dimensional operators in  $\mathcal{H}$  acting in accordance with the formula  $\psi\psi^*\varphi = (\varphi|\psi)\psi$ . The space  $\mathcal{L}$  generated by the operators  $\psi\psi^*$  is the dual to the algebra of self-adjoint operators in  $\mathcal{H}$  with respect to the bilinear form  $\langle \rho, A \rangle$  generated by the form  $(A\psi|\psi)$  on the elements  $\rho = \psi\psi^*$ . We consider (2.3) with the commutator Poisson bracket  $\{\cdot, \cdot\} = i[\cdot, \cdot]$  and the self-adjoint Vlasov Hamiltonian  $H(t, \rho) = H(t, \rho)^*$ . On the manifold  $M = \{\psi\psi^* : \psi \in \mathcal{H}\} \subset \mathcal{L}$  of the one-dimensional operators  $\rho = \psi\psi^*$  it takes the form

$$(2.6) \quad i \frac{\partial \psi(t)\psi(t)^*}{\partial t} = H(t, \psi(t)\psi(t)^*)\psi(t)\psi(t)^* - \psi(t)\psi(t)^*H(t, \psi(t)\psi(t)^*),$$

which decomposes into a pair of mutually adjoint equations in  $\mathcal{H}$  – the abstract Hartree equations

$$(2.7) \quad i \frac{\partial \psi(t)}{\partial t} = H(t, \psi(t)\psi(t)^*)\psi(t), \quad -i \frac{\partial \psi(t)^*}{\partial t} = \psi(t)^*H(t, \psi(t)\psi(t)^*).$$

This generalizes the Schrödinger equation for the *Hartree Hamilton operators*  $H(t, \psi\psi^*)$ , the Vlasov Hamiltonians on  $M$  that depend on one-dimensional density  $\psi\psi^*$  in the space  $\mathcal{H}$ . Note that the equations (2.6) describe a classical field-Hamiltonian system on the space  $\mathcal{H}$  with respect to the symplectic form  $2 \operatorname{Im}(\varphi|\psi)$ , which determines Poisson brackets on the submanifold  $M \subset \mathcal{L}$ :

$$(2.8) \quad \{\gamma, \alpha\}_{cl}(\psi\psi^*) = 2 \operatorname{Im}(H(\psi\psi^*)\psi|A(\psi\psi^*)\psi),$$

where  $A(\psi\psi^*) = \delta\alpha(\psi\psi^*)$ .

2.3. A solution of equation (2.7) can be written down implicitly for given initial condition  $\psi(t_0) = \varphi$  by means of a time ordered mapping [5]:

$$(2.9) \quad \psi(t) = \exp \left\{ -i \int_{t_0}^t H(s, \psi(s)\psi(s)^*) ds \right\} \varphi,$$

which generalizes the ordinary Feynman time ordered product. Thus, the solution of (2.3) reduces in the given case to “disentangling” the expression (2.9), i.e., to finding an operator  $V(t, t_0, \varphi\varphi^*)$  by means of which (2.9) can be represented explicitly as a function of  $\varphi$ :

$$(2.10) \quad \psi(t) = V(t, t_0, \varphi\varphi^*)\varphi$$

Sometimes, this can be readily done only for certain conditions  $\varphi$ . For example, if for every  $\nu = \varphi^*\varphi$  there exists a vector  $\varphi = \varphi_\nu$  such that

$$(2.11) \quad H(s, \varphi_\nu \varphi_\nu^*) \varphi_\nu = \omega(s, \nu) \varphi_\nu$$

directly for all  $t$ , then the solution (2.10) has the form

$$(2.12) \quad \psi(t) = \exp \left\{ -i \int_{t_0}^t \omega(s, \nu) ds \right\} \varphi_\nu.$$

This fact is a special case of the following assertion.

**Proposition 2.** Suppose Equation. (2.7) has one or several integrals  $\pi_j(\psi\psi^*)$  of the form  $\pi_j(\psi\psi^*) = \psi^*P_j\psi$ , where  $P_j$  are self-adjoint commuting operators in  $\mathcal{H}$ , and suppose there exists an extremal  $\psi = \varphi_{\mathbf{p}}$ ,  $\mathbf{p} = \{p_j\}$  of the functional  $\gamma(t, \psi\psi^*)$  which is the same for all  $t$  and satisfies the conditions  $\varphi_{\mathbf{p}}^*P_j\varphi_{\mathbf{p}} = p_j$ ,

$$(2.13) \quad H(t, \varphi_{\mathbf{p}}\varphi_{\mathbf{p}}^*)\varphi_{\mathbf{p}} = \sum_j \nu^j P_j\varphi_{\mathbf{p}},$$

where  $\nu^j$  are parameters conjugate to  $p_j$  (Lagrangian multipliers) determined by the conditions

$$(2.14) \quad \nu^j(t, \mathbf{p}) = \frac{\partial h(t, \mathbf{p})}{\partial p_j}, \quad h(t, \mathbf{p}) = \gamma(t, \varphi_{\mathbf{p}}\varphi_{\mathbf{p}}^*).$$

Then the solution of Equation. (2.7) with the initial condition  $\psi(0) = \varphi_{\mathbf{p}}$  has the form

$$(2.15) \quad \psi(t) = \exp \left\{ -i \int_{t_0}^t \mathbf{v}(s, p) \mathbf{P} ds \right\} \varphi_{\mathbf{p}},$$

where  $\mathbf{v}\mathbf{P} = \sum \nu^j P_j$ .

This can be proved by direct substitution of (2.15) into (2.3) with allowance for the invariance of  $\gamma$  under the transformations  $\psi \mapsto e^{iq\mathbf{P}}\psi$  for all real parameters  $\mathbf{q} = \{q^j\}$ . The parameters  $q^j$ ,  $\nu^j$ ,  $p_j$  have the meaning of generalized coordinates, velocities and momenta respectively.

**Definition 1.** The solution (2.15) of Equation. (2.7) is called a generalized soliton with momenta  $p_j$ .

To the soliton (2.12) there obviously corresponds the case of existence of an extremal  $\varphi_\nu$  for the integral  $\pi_0(\psi\psi^*) = \psi^*\psi$  generated by the identity operator  $P_0 = I$ .

Suppose, for example, the space  $\mathcal{H}$  is the space of functions  $\psi(\mathbf{x})$  on  $\mathbb{R}^d \ni \mathbf{x}$  with scalar product  $(\varphi|\psi) = \int \varphi(\mathbf{x}) \overline{\psi}(\mathbf{x}) d\mathbf{x}$ , and the functional  $\gamma$  is invariant with respect to the group of displacements:

$$\gamma(t, \psi_{\mathbf{q}}\psi_{\mathbf{q}}^*) = \gamma(t, \psi\psi^*) \quad \forall \psi_{\mathbf{q}}(\mathbf{x}) = \psi(\mathbf{x} - \mathbf{q}), \mathbf{q} \in \mathbb{R}^d.$$

In this case, the conditions of proposition (2) are satisfied by the ordinary momenta

$$\pi_j(\psi\psi^*) = -i \int \overline{\psi}(\mathbf{x}) \partial_j \psi(\mathbf{x}) d\mathbf{x}, \quad j = 1, \dots, d,$$

and one can expect that ordinary soliton solutions would exist - travelling waves with momenta  $\mathbf{p} = \{p_j\}$ :

$$\psi(\mathbf{x}, t) = \varphi \left( \mathbf{x} - \int_{t_0}^t \mathbf{v}(s, \mathbf{p}) ds \right).$$

This last holds, for example, for the ordinary (scalar) Hartree-Fock equation, for which the Hamilton operator  $H(t, \psi\psi^*)$  has the form

$$(2.16) \quad H(\psi\psi^*) = -\partial^2 + \int \omega(\mathbf{x} - \mathbf{x}') |\psi(\mathbf{x}')|^2 d\mathbf{x}',$$

where  $\partial^2$  is the Laplacian with respect to  $\mathbf{x} \in \mathbb{R}^d$ , and where  $\omega(\mathbf{x} - \mathbf{x}')$  is the potential of the two-body interaction, for example, the Green's function for the equation  $\partial^2 u(\mathbf{x}) = 4\pi |\psi(\mathbf{x})|^2$ . Our treatment also encompasses more general cases of Equation. (2.7), for example, if  $\partial^2$  is a square with respect to some indefinite metric in  $\mathbb{R}^d$ . An interesting case arises when  $\mathbb{R}^d$  is four-dimensional space-time, for which  $\partial^2$  is the d'Alembert operator  $\square$ .

2.4. In the general case, if there is a discrete basis in  $\mathcal{H}$ , it is convenient to identify the vectors  $\psi^*$  and  $\varphi$  with covariant and contravariant tensors of the first rank  $\psi_x$ ,  $\varphi^x = \overline{\varphi_x}$ , fixing this basis. In this notation, the form  $\psi^*\varphi = (\varphi|\psi)$  is simply the tensor contraction  $\psi_x\varphi^x$  such that the mapping  $\psi \mapsto \psi^*$  is the operation of complex conjugation and lowering the superscript by means of an antilinear metric tensor  $J : \psi_x = J_{xy}\psi^y$ ,  $\psi^x = J^{xy}\psi_y$ . At the same time, the operators  $\rho \in \mathcal{L}$ ,  $A \in \mathcal{A}$  are represented by matrices  $\rho_y^x$ ,  $A_x^y$ , the bilinear form  $\langle \rho, A \rangle$  is the contraction  $\rho_y^x A_x^y$ , and the derivative  $H(\rho) = \delta\gamma(\rho)$  is given by the partial derivatives  $H_x^y(\rho) = \frac{\partial\gamma(\rho)}{\partial\rho_y^x}$ . Further, we shall encounter tensor powers  $\rho^{\otimes n}$ ,  $\delta^{\otimes n}$ , which in our notation are the products  $\rho_{y_1}^{x_1} \dots \rho_{y_n}^{x_n}$ ,  $\delta_{x_1}^{y_1} \dots \delta_{x_n}^{y_n} = \frac{\partial^n}{\partial\rho_{y_1}^{x_1} \dots \partial\rho_{x_n}^{y_n}}$ . Formally, the tensor notation can also be used even in the continuous index case if the partial derivatives are understood as variational derivatives and contraction as an integral with respect to a given measure; however the matrix elements are then represented by generalized functions.

### 3. UNIFORMIZATION AND SECOND QUANTIZATION

3.1. We shall say that a Hamiltonian system (2.3) is uniformizable if the Hamilton functional  $\gamma(t, \rho)$  is a polynomial:

$$(3.1) \quad \gamma(t, \rho) = \sum_{n=1}^N \frac{1}{n!} \langle \rho^{\otimes n}, W^{(n)}(t) \rangle$$

or entire function ( $N = \infty$ ). Here,  $\rho^{\otimes n}$  are tensor powers of the element  $\rho \in \mathcal{L}$  generating the symmetrical tensor-spaces  $\mathcal{L}_n$ <sup>3</sup>,  $W^{(n)}(t)$  are elements of the dual algebras  $\mathcal{A}^{(n)} = \mathcal{L}_n^*$ , and  $\langle \rho^{\otimes n}, A^{(n)} \rangle$  are linear forms on  $\mathcal{A}^{(n)} \ni A^{(n)}$  equal to the products  $\langle \rho, A_1 \rangle \dots \langle \rho, A_n \rangle$  on the  $\mathcal{A}^{(n)}$ -generating elements  $A^{(n)} = A_1 \otimes \dots \otimes A_n \equiv A_1^{\otimes n}$ . Note that  $W^{(n)}$  are represented as the symmetric tensors in any discrete basis, and are uniquely determined from  $\gamma(\rho)$  as  $n$ -th

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<sup>3</sup>That is  $\mathcal{L}_n$  is the minimal Fréchet subspace of the projective tensor power  $\mathcal{L}^{\otimes n}$  containing all  $\rho^{\otimes n}$ ,  $\rho \in \mathcal{L}$ .

derivatives  $\delta^{(n)}\gamma(\rho)$  at the point  $\rho = 0$  defined as the tensor power  $\delta^{(n)} \equiv \delta^{\otimes n}$  of the derivative  $\delta$  in (2.1) such that

$$\delta^{\otimes n} e^{\langle \rho, A \rangle} = e^{\langle \rho, A \rangle} A^{\otimes n}$$

for any  $A \in \mathcal{A}$ . The symmetrical spaces  $\mathcal{A}^{(n)}$  like the full tensor power spaces  $\mathcal{A}^{\otimes n}$  are Hamiltonian (Lie-Jordan) algebras with respect to products  $H^{(n)} \cdot A^{(n)}$ ,  $\{H^{(n)}, A^{(n)}\}$  defined on the primitive elements  $H^{(n)} = \bigotimes_{i=1}^n H_i$ ,  $A^{(n)} = \bigotimes_{i=1}^n A_i$  as

$$(3.2) \quad H^{(n)} \cdot A^{(n)} = \bigotimes_{i=1}^n H_i \cdot A_i, \quad \{H^{(n)}, A^{(n)}\} = \sum_{j=1}^n \bigotimes_{\substack{i=1 \\ i \neq j}}^n H_i \cdot A_i \otimes \{H_j, A_j\}$$

such that  $H_1^{\otimes n} \cdot A_1^{\otimes n} = (H_1 A_1)^{\otimes n}$ ,  $\{H_1^{\otimes n}, A_1^{\otimes n}\} = n H_1^{\otimes(n-1)} \cdot A_1^{\otimes(n-1)} \otimes_s \{H_1, A_1\}$ .

The symmetry of  $W^{(n)}(t)$  makes it possible to write the Vlasov Hamiltonian  $H(\rho) = \delta\gamma(\rho)$  determining the Vlasov equation (2.3), in the form

$$(3.3) \quad H(t, \rho) = \sum_{n=1}^{N-1} \frac{1}{n!} \left\langle \rho^{\otimes n}, W^{(n+1)}(t) \right\rangle,$$

where  $\langle \rho^{\otimes n}, A^{(n+1)} \rangle$ , the elements of the algebra  $\mathcal{A}$ , are equal to  $\langle \rho, A_1 \rangle^n A$  for  $A^{(n+1)} = A_1 \otimes \dots \otimes A_1$ . An example of such an expression with  $N = 2$  gives the operator (2.16) if  $\psi\psi^*$  is replaced by an arbitrary density operator  $\rho = \sum \lambda_i \psi_i \psi_i^*$ .

**Definition 2.** Let  $\varepsilon > 0$  be a parameter; an  $\varepsilon$ -uniformization of the classical Hamiltonian system (3.3) with the Hamiltonian functional  $\gamma(t, \rho)$  is the system of the Hamiltonians

$$(3.4) \quad H^{(n)}(t) = \frac{1}{\varepsilon} (1 + \delta\varepsilon)^{\otimes n} \gamma(t, \rho) |_{\rho=0} = \sum_{m=1}^n \binom{n}{m} \varepsilon^{m-1} \left( I^{(n-m)} \otimes_s W^{(m)}(t) \right)$$

as the real elements in the Hamiltonian algebras  $\mathcal{A}^{(n)}$ . Here  $I^{(n-m)} \otimes_s W^{(m)}$  is the symmetrization of the tensor products  $I^{(n-m)} \otimes W^{(m)}$  of the identities  $I^{(n-m)} = I \otimes \dots \otimes I$  of the algebras  $\mathcal{A}^{(n-m)}$  and the elements  $W^{(m)} \in \mathcal{A}^{(m)}$ .

For example, the uniformization of the Hamilton operators (2.16) of the Hartree equation (2.7) leads to the system of operators

$$(3.5) \quad H^{(n)} = - \sum_{i=1}^n \partial_i^2 + \varepsilon \sum_{i_1=1}^n \sum_{i_2=1}^{i_1-1} \omega(\mathbf{x}_{i_1} - \mathbf{x}_{i_2}),$$

where  $\partial_i^2$  are the Laplace operators with respect to the variables  $\mathbf{x}_i \in \mathbb{R}_i^d$ . For four-dimensional space-time  $\mathbb{R}^4$ , for which  $\partial^2$  is the d'Alembert operator, the uniformized Hamiltonians are ultrahyperbolic operators for  $n \geq 2$ .

3.2. Note that the Hamiltonians (3.4) determine in the spaces  $\mathcal{L}_n$  the linear equations

$$(3.6) \quad \frac{\partial \rho_n(t)}{\partial t} = \left\{ \rho_n(t), H^{(n)}(t) \right\}$$

which are abstract Liouville (or von Neumann) equations understood as predual to the abstract Heisenberg equations

$$(3.7) \quad \frac{\partial A^{(n)}(t)}{\partial t} = \left\{ H^{(n)}(t), A^{(n)}(t) \right\},$$

so that the relation  $\langle \rho_n(t), A_0^{(n)} \rangle = \langle \rho_n^0, A^{(n)}(t) \rangle$  holds for all initial  $A_0^{(n)} = A^{(n)}(t_0) \in \mathcal{A}^{(n)}$  and  $\rho_n^0 = \rho_n(t_0) \in \mathcal{L}_n$ .

In order to develop a compact functional derivative representation of all these independent equations and find their relation to the abstract Vlasov equation we introduce the representing functionals

$$(3.8) \quad \alpha(\rho) = \exp \left\{ -\frac{1}{\varepsilon} \langle \rho, I \rangle \right\} \sum \frac{1}{n! \varepsilon^{n-1}} \langle \rho^{\otimes n}, A^{(n)} \rangle,$$

by means of which the elements  $A^{(n)} \in \mathcal{A}^{(n)}$  can be calculated as the derivatives

$$(3.9) \quad A^{(n)} = \varepsilon^{n-1} \delta^{\otimes n} \exp \left\{ \frac{1}{\varepsilon} \langle \rho, I \rangle \right\} \alpha(\rho) |_{\rho=0} = \frac{1}{\varepsilon} (1 + \varepsilon \delta)^{\otimes n} \alpha(\rho) |_{\rho=0}.$$

Here,  $A^{(0)}$  is a complex number  $A^{(0)} \in \mathbb{C} \equiv \mathcal{A}^{(0)}$ .

**Proposition 3.** *The system of equations (3.7) can be expressed in the form of the pseudodifferential equation*

$$(3.10) \quad \frac{\partial \alpha(t, \rho)}{\partial t} = \left\langle \left\{ \frac{2}{\rho}, H \left( t, \frac{2}{\rho} \cdot \left( I + \varepsilon \delta \right) \right) \right\}, \delta \right\rangle \alpha(t, \rho),$$

defined in terms of the symbol-Hamiltonian

$$H(t, \rho \cdot (I + \varepsilon A)) \equiv \exp \left\{ \varepsilon \left\langle \frac{2}{\rho}, \delta \cdot A \right\rangle \right\} H(t, \rho),$$

where the superscripts indicate the order of application of the operators of (tensor) multiplication by  $\rho$  and the derivation with respect to  $\rho$ . The equation (3.10) has the limit at  $\varepsilon \rightarrow 0$  of first order differential equation

$$\frac{\partial}{\partial t} \alpha(t, \rho) = \langle \{\rho, H(t, \rho)\}, \delta \rangle \alpha(t, \rho)$$

giving the weak form of the Vlasov equation for  $\alpha(t, \rho) = \langle \rho(t), A \rangle$ .

The proof of this proposition is contained in the derivation of the formula (expansions) (3.13) and (3.14) in the Appendix. Here we only note that the symbol

$$G(t, \rho, A) = \langle \{\rho, H(t, \rho \cdot (I + \varepsilon A))\}, A \rangle$$

of the equation (3.10), which is determined on the exponential functionals  $\alpha(\rho) = \exp \langle \rho, A \rangle$  by

$$(3.11) \quad G\left(t, \frac{2}{\rho}, \frac{1}{\delta}\right) e^{\langle \rho, A \rangle} = e^{\langle \rho, A \rangle} G(t, \rho, A),$$

depends analytically on  $\varepsilon$  as a power series expansion in  $\varepsilon$ . Comparing (3.10) with (2.4), we readily see that for  $\varepsilon = 0$  the derivative (3.10) simply coincides with the derivative of  $\alpha(t, \rho) = \alpha(\rho(t))$  along the trajectories of the Vlasov equation (2.3). Expanding the operator  $G(t, \rho, \delta)$  in powers of  $\varepsilon$  and substituting in Equation (3.10) the solution in the form of the series  $\alpha(t, \rho) = \sum_{k=0}^{\infty} \varepsilon^k \alpha_k(t, \rho)$ , we obtain by the method of successive approximation the following system of coupled equations:

$$(3.12) \quad \frac{\partial \alpha_k(t)}{\partial t} = \sum_{n=0}^k \frac{1}{(n+1)!} \left\langle \left\{ \left( \frac{2}{\rho} \right)^{\otimes(n+1)}, H^{(n+1)}(t, \rho) \right\}, \left( \frac{1}{\delta} \right)^{\otimes(n+1)} \right\rangle \alpha_{k-n}(t),$$

where  $H^{(m)}(t, \rho) = \delta^{\otimes m} \gamma(t, \rho)$ . Thus, the Vlasov equation (2.3) is a Hamiltonian system of the characteristics for the zeroth-order approximation  $\alpha_0(t)$  with initial condition  $\alpha(t_0) = \alpha_0$ .

**3.3.** In general, the uniformization procedure can be regarded as a linear functor from the classical algebra of entire Hamilton functions  $\gamma(\rho)$  with ordinary product  $\gamma(\rho)\alpha(\rho)$  and Poisson bracket (2.5) to a new Lie-Jordan (Hamiltonian) algebra  $\mathfrak{A}$  of these functions with not necessarily associative product

$$(3.13) \quad \begin{aligned} (\gamma \cdot \alpha)(\rho) &= \frac{1}{\varepsilon} \gamma \left( \rho \cdot \left( I + \varepsilon \delta \right) \right) \alpha(\rho) \\ &= \sum_{n=0}^{\infty} \frac{\varepsilon^{n-1}}{n!} \langle \rho^{\otimes n}, \delta^{\otimes n} \gamma(\rho) \cdot \delta^{\otimes n} \alpha(\rho) \rangle \end{aligned}$$

and uniformized Poisson brackets

$$(3.14) \quad \begin{aligned} \{\gamma, \alpha\}(\rho) &= \langle \rho, \{\delta \gamma(\rho \cdot (I + \varepsilon \delta)), \delta \alpha(\rho)\} \rangle \\ &= \sum_{n=0}^{\infty} \frac{\varepsilon^{n-1}}{n!} \langle \rho^{\otimes n}, \{\delta^{\otimes n} \gamma(\rho), \delta^{\otimes n} \alpha(\rho)\} \rangle. \end{aligned}$$

The latter in accordance with (2.2) determines for a fixed Hamilton functional  $\gamma(t, \rho)$  the *uniformized Hamiltonian derivation* (3.10) on the algebra  $\mathfrak{A}$ , in which  $H(t, \rho) = \delta \gamma(t, \rho)$ , since Equations (3.8) and (3.9) establish an isomorphism between the algebra  $\mathfrak{A}$  and the direct sum algebra  $\bigoplus_{n=0}^{\infty} \mathcal{A}^{(n)}$  of symmetrized tensor powers  $\mathcal{A}^{(n)}$  of the algebra of elementary observables  $\mathcal{A}$  since the products (3.13) and (3.14) are simply representing functionals for the products  $H^{(n)} \cdot A^{(n)}$ ,  $\{H^{(n)}, A^{(n)}\}$  of components of the elements  $\hat{A} = \bigoplus_{n=0}^{\infty} A^{(n)}$  and  $\hat{H} = \bigoplus_{n=0}^{\infty} H^{(n)}$  (this last follows from the proof of Proposition (1)). In the limit  $\varepsilon \mapsto 0$ , the Jordan product (3.13) multiplied

by  $\varepsilon$  goes over into the ordinary pointwise product  $\gamma(\rho)\alpha(\rho)$ , and the Lie product (3.14) in this limit obviously coincides with the classical Poisson brackets (2.5).

In the case when the Jordan product in  $\mathcal{A}$  is associative, the uniformized product (3.13) also remains associative. This is the case when  $\mathcal{A}$  is the classical algebra of single particle observables  $A(\mathbf{q}, \mathbf{p})$ . Then the uniformization is equivalent to the transition from the classical field Hamiltonian system described by the Vlasov equation itself to the classical Hamiltonians (3.2), where  $\varepsilon$  is the coupling constant. In this sense, the uniformization procedure is the inverse of the procedure of derivation of the Vlasov equation from the Bogolyubov equation [6]. One could naturally call this replacement of the classical field system by a system with variable particle number as quantization, however, we shall avoid this terminology here since by quantization one always understands transitions from a classical to a definitely nonclassical system, whereas the example considered here shows that the uniformized system may remain classical.

The same interpretation remains valid in the quantum case when  $\mathcal{A}$  is an algebra of operators in Hilbert single-particle space  $\mathcal{H}$ . The uniformized system of a variable number of identical quantized particles is, in contrast to the original field classical system, no longer classical, and in this sense the described procedure can be regarded as a certain quantization, and it can be given a Heisenberg form with generating commutation relations

$$(3.15) \quad [\langle \rho, H \rangle, \langle \rho, A \rangle] = \langle \rho, [H, A] \rangle,$$

if one introduces an associative and noncommutative product  $(\gamma \cdot \alpha)(\rho)$  as the representing functional for the products  $H^{(n)} \cdot A^{(n)}$  of operators in the tensor powers  $\mathcal{H}^{(n)}$  of the Hilbert space  $\mathcal{H}$ . At the same time, the complex extension of  $\mathcal{A}$  is a linear noncommutative associative algebra isomorphic to the algebra of operators  $\hat{A} \in \mathfrak{H} = \bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}$  of decomposed form  $\hat{A} = \bigoplus A^{(n)}$ , where  $A^{(n)}$  are symmetric tensor operators in  $\mathcal{H}^{(n)}$  that leave invariant the subspaces  $\mathcal{H}_{\pm}^{(n)} \subset \mathcal{H}^{(n)}$  of the completely symmetric  $(\mathcal{H}_{+}^{(n)})$ , as well as the completely antisymmetric  $(\mathcal{H}_{-}^{(n)})$  tensors  $\psi^{(n)} \in \mathcal{H}^{(n)}$ . This last means that the elements of the algebra  $\mathcal{A}$  can be interpreted as the observables of a second quantized system of identical bosons or fermions depending on whether or not they are regarded as operators  $\hat{A}$  in Fock spaces  $\mathfrak{H}_{\pm} = \bigoplus_{n=0}^{\infty} \mathcal{H}_{\pm}^{(n)}$  that are symmetric or antisymmetric under permutations of particles. Such Fock representations of the algebra  $\mathcal{A}$  are obtained by means of the functionals (3.7) in accordance with the formula  $A = \frac{1}{\varepsilon} \alpha \left( \varepsilon a_{\pm}^1 a_{\pm}^2 \right)$ , where  $\hat{a}_{\pm}$ ,  $\hat{a}_{\pm}^*$  are vector operators of annihilation and creation of bosons (with plus sign) and fermions (with minus sign) acting on the Fock spaces  $\mathfrak{H}_+$ ,  $\mathfrak{H}_-$  respectively, and the numbers 1 and 2 above the operators indicate the normal order, in accordance with which the monomials

$\left\langle \begin{pmatrix} 1 & 2 \\ aa^* & \end{pmatrix}^{\otimes n}, A^{(n)} \right\rangle$  are written as the normal products  $\hat{a}^{*\otimes n} A^{(n)} \hat{a}^{\otimes n}$ . The linear observables  $\alpha(\rho) = \langle \rho, A \rangle$  as functions on  $\mathcal{A} \ni A$  are then represented in  $\mathfrak{H}_\pm$  by operator-valued forms  $\hat{n}(A) = \hat{a}_\pm^* \hat{A} \hat{a}_\pm$ , and the commutation relations (3.15) can be deduced directly from the canonical commutation relations for the operators  $\hat{a}_\pm, \hat{a}_\pm^*$ :

$$(3.16) \quad [\hat{n}(H), \hat{n}(A)] = [\hat{a}_\pm^* H \hat{a}_\pm, \hat{a}_\pm^* A \hat{a}_\pm] = \hat{a}_\pm^* [H, A] \hat{a}_\pm = \hat{n}([H, A]).$$

Hamiltonians  $\hat{H} = \frac{1}{\varepsilon} \gamma \begin{pmatrix} 1 & 2 \\ \varepsilon aa^* & \end{pmatrix}$  of the nonlinear form (3.1) can be represented in the Fock spaces  $\mathfrak{H}_\pm$  by expansions in the coupling constant  $\varepsilon$ :

$$(3.17) \quad \hat{H}(t) = \sum_{n=1}^N \frac{\varepsilon^{n-1}}{n!} \hat{a}_\pm^{*\otimes n} W^{(n)}(t) \hat{a}_\pm^{\otimes n}.$$

Thus, uniformization of the quantum Vlasov equation can be regarded as the algebraic basis for a simplified method of second quantization which avoids explicit introduction of unphysical - odd - observables (which do not commute with the operator of the total particle number  $n(\hat{\mathbf{I}})$ ) and takes into account in a unified manner for Bose and Fermi systems the principle of indistinguishability of identical particles. The calculus (3.13-3.14) of the representing functionals (3.8) together with Equation. (3.9) combines the holomorphic calculus of the operators  $\hat{A} = \bigoplus A^{(n)}$  in the fock spaces  $\mathfrak{H}_\pm$  developed by Bargmann [10] for the case of Bose-Einstein statistics and Berezin [11] for Fermi-Dirac statistics; at the same time, the commutation relations (3.15) replace the canonical commutation relations for the operators  $\hat{a}_\pm, \hat{a}_\pm^*$ .

All that we have said remains true in the more general case when the metric of the space  $\mathcal{H}$  can be indefinite except for the interpretation of uniformization as second quantization since the spaces  $\mathfrak{H} = \bigoplus \mathcal{H}^{(n)}$ , like their subspaces  $\mathfrak{H}_\pm$  need not be Hilbert spaces, but only, like  $\mathcal{H}$ , pseudo-Hilbert spaces. The representations  $\bigoplus \mathcal{A}^{(n)}$  of the algebra  $\mathcal{A}$  by the operators (3.8) in the spaces  $\mathcal{H}^{(n)}$  are then representations with indefinite metric, and are not Fock representations. For this reason, the dual elements  $\rho = \bigotimes_n \rho_n$ , where  $\rho_n \in \mathcal{L}_n$  are operators of the form  $\rho_n = \sum_i \lambda_i \psi_i^{(n)} \psi_i^{(n)*}$ , cannot, in general, be interpreted as statistical operators of  $n$ -particle states even if  $\lambda_i \geq 0$  and  $\sum_i \lambda_i = 1$ . Nevertheless, investigation of the linear equations (3.4) or the (abstract) Schrödinger equations

$$(3.18) \quad i \frac{\partial \psi^{(n)}(t)}{\partial t} = H^{(n)} \psi^{(n)}(t), \quad -i \frac{\partial \psi^{(n)}(t)^*}{\partial t} = \psi^{(n)}(t)^* H^{(n)},$$

into which they decompose for initial conditions of the one-dimensional form  $\rho_n(t_0) = \psi^{(n)} \psi^{(n)*}$ , enables one to extract the entire information about the dynamics of the uniformized system. It is sufficient to consider only the symmetric or antisymmetric  $n$ -tensors  $\psi^{(n)} \in \mathcal{H}^{(n)}$ , for which the products

$\psi^{(n)}\psi^{(n)*}$  are elements of the space  $\mathcal{L}_n$  of symmetric  $n$ -tensors  $\rho_n$  since equations (3.13) preserve the symmetry of  $\psi^{(n)}$  by virtue of the fact that the Hamiltonians  $H^{(n)} \in \mathcal{A}^{(n)}$  commute with the permutation operators.

#### 4. UNIFORMIZED SOLUTIONS OF THE HARTREE EQUATION

4.1. Here, we consider a method based on uniformization for disentangling the Time ordered mapping (2.9), this giving solutions of the Hartree equation (2.7) with Hamiltonian of the form (3.3), where  $\rho = \psi\psi^*$  and  $\langle\rho^{\otimes n}, W^{(n+1)}\rangle = \psi^{*\otimes n}W^{(n+1)}\psi^{\otimes n}$ . To be specific, we shall assume throughout that the forms  $\psi^{*\otimes n}A^{(n)}\psi^{\otimes n}$  are c-numbers, although they could also be regarded as even elements of a Grassmann algebra with  $\psi$  and  $\psi^*$  regarded as anticommuting generators. At the same time,  $\psi^{*\otimes n}A^{(n)}\psi^{\otimes n} = \psi^*A_1\psi \cdots \psi^*A_n\psi$  for  $A^{(n)} = \bigoplus_{i=1}^{\infty} A_i$ , and  $\psi^*A\psi = (A\varphi|\psi)$ .

We shall regard the differences  $H^{(n+1)}(t) - H^{(n)}(t) \otimes \mathbf{I}$  as operators in  $\mathcal{H}_+^{(n)} \otimes \mathcal{H}$  and denoted by  $\Omega^{(n+1)}(t, t_0)$  the operators in  $\mathcal{H}_+^{(n)} \otimes \mathcal{H}$  obtained by there transformation:

$$(4.1) \quad \Omega^{(n+1)}(t, t_0) = U^{(n)}(t, t_0)^* \left( H^{(n+1)}(t) - H^{(n)}(t) \otimes \mathbf{I} \right) U^{(n)}(t, t_0),$$

where  $U^{(n)}(t, t_0)$  are unitary operators in  $\mathcal{H}_+^{(n)}$  determining the solutions of equations (3.4):

$$(4.2) \quad U^{(n)}(t, t_0) = \exp \left\{ -i \int_{t_0}^t H^{(n)}(s) ds \right\}.$$

**Definition 3.** We give the name  $\varepsilon$  solution of equation (2.7) with the operator (3.3) and initial condition  $\psi(t_0) = \psi$  to the following expression  $\psi(t) = V(t, t_0, \varphi\varphi^*)\varphi$ , where

$$(4.3) \quad V(t, t_0, \varphi\varphi^*)\varphi = \exp \left\{ -\frac{1}{\varepsilon} \varphi^* \varphi \right\} \sum_{n=0}^{\infty} \frac{\varepsilon^{n-1}}{n!} \varphi^{*\otimes n} V^{(n+1)}(t, t_0) \varphi^{\otimes n},$$

and  $V^{(n+1)}(t, t_0)$  are unitary operators in  $\mathcal{H}_+^{(n)} \otimes \mathcal{H}$  with generators (4.1), i.e., the time-ordered products

$$(4.4) \quad V^{(n+1)}(t, t_0) = \exp \left\{ -i \int_{t_0}^t \Omega^{(n+1)}(s, t_0) ds \right\}.$$

The justification for this definition is the following proposition, which reduces the problem of disentangling the Time-ordered mapping (2.9) for uniformized systems to calculation of Time-ordered products (4.4).

**Proposition 4.** The solution of the uniformized equation (2.7) with any initial condition  $\psi(t_0) = \varphi$  coincides with the  $\varepsilon \rightarrow 0$  limit of the corresponding  $\varepsilon$  solution.

This proposition is readily proved by noting that the Time-ordered products (4.4) are found by disentangling the operator  $(U^{(n)} \otimes \mathbf{I})^{-1}$  with the

generator  $H^{(n)} \otimes \mathbf{I}$  from the operator  $U^{(n+1)}$  with generator  $H^{(n+1)} = H^{(n)} \otimes \mathbf{I} + \Omega^{(n+1)}$ ,

$$(4.5) \quad V^{(n+1)}(t, t_0) = \left( U^{(n)}(t, t_0) \otimes \mathbf{I} \right)^{-1} U^{(n+1)}(t, t_0).$$

This, the  $\varepsilon$  solution essentially reduces to finding the unitary operators (4.2) representing the solutions of the Schrödinger equation (3.18) in the form  $\psi^{(n)}(t) = U^{(n)}(t, t_0)\varphi^{(n)}$  for any initial condition  $\varphi^{(n)} \in \mathcal{H}_+^{(n)}$ . The operators (4.5) multiplied by  $\sqrt{n}$  are none other than the nonzero matrix elements of the vector operators of annihilation of bosons  $\hat{a}(t)$  in the Heisenberg representation, and the  $\varepsilon$  solution as a function of  $\varphi$ ,  $\varphi^*$  is the symbol of the vector operator  $\sqrt{\varepsilon}\hat{a}(t) = \hat{\psi}(t)$  ordered normally with respect to the initial operators  $\hat{\varphi} = \sqrt{\varepsilon}\hat{a}_+(t_0)$ . For  $\varepsilon = 0$ , the initial vector operators  $\hat{\varphi}$  coincide with the numerical vectors  $\varphi$ , and the  $\varepsilon$  solutions coincide with the solutions of the Hartree equation since the Heisenberg equation for the symbol  $\hat{\psi}(t)$  in the representation of the instantaneous time  $t$  coincides exactly with the Hartree equation.

4.2. The disentangling of the Time-ordered mappings in this way requires the solution of all the Schrödinger equations (3.18) for  $n = 1, 2, \dots$ . It is of interest to ask whether one can find not all but at least some solutions of the Hartree equation by solving not all but, for example, two neighboring equations (3.18).

Let  $\varphi^{(n)}$ ,  $\varphi^{(n+1)}$  be (generalized) stationary solutions of Equations. (3.18) corresponding to the eigenvalues  $\lambda_n(t)$ ,  $\lambda_{n+1}(t)$  of the uniformized Hamiltonians  $H^{(n)}(t)$ ,  $H^{(n+1)}(t)$ . One can evidently write the matrix element  $\psi(t) = \varphi^{(n)*}V^{(n+1)}(t, t_0)\varphi^{(n+1)}$  of the operator (4.5) in the form (2.12), where

$$(4.6) \quad \varphi^{(n)*}\varphi^{(n+1)} = \varphi_v, \quad v = \varepsilon n, \quad \omega(t, v) = \lambda_{n+1}(t) - \lambda_n(t).$$

If all quantities (4.6) have a limit as  $\varepsilon \rightarrow 0$ , then one can expect that in this limit the given matrix element defines a certain stationary solution of the Hartree equation, while for  $\varepsilon \neq 0$  it gives an approximate solution of this equation.

Generally speaking, suppose the Hartree equation (2.7) has one or several integrals of quadratic form  $\varphi^*P_j\varphi$ , including  $\mathbf{P}_0 = \mathbf{I}$ , and suppose  $|\mathbf{p}\rangle$  are (generalized) eigenvectors of the operators  $\varepsilon\hat{a}^*\hat{P}_j\hat{a}$  and the Hamilton operator  $\gamma\left(\varepsilon\hat{a}\hat{a}^*\right)$  in the space  $\mathfrak{H} = \bigoplus \mathcal{H}_+^{(n)}$  corresponding to the eigenvalues  $p_j$  and  $h(t, \mathbf{p})$ . We denote the integrals of the matrix elements of the Heisenberg vector operators  $\hat{a}(t)$  by

$$(4.7) \quad \psi_{\mathbf{p}}(t) = \sqrt{\varepsilon} \int \langle \mathbf{p} | \hat{a}(t) | \mathbf{p}' \rangle d\mathbf{p}'.$$

**Proposition 5.** *The vectors (4.7) can be expressed in terms of the initial vectors  $\varphi_{\mathbf{p}} = \psi(t_0)$  in accordance with*

$$(4.8) \quad \psi_{\mathbf{p}}(t) = \exp \left\{ -\frac{i}{\varepsilon} \int_{t_0}^t (h(s, \mathbf{p} + \varepsilon \mathbf{P}) - h(s, \mathbf{p})) ds \right\} \varphi_{\mathbf{p}}.$$

*Proof.* For any Heisenberg operator  $\hat{A}(t)$  we introduce in the space  $\mathfrak{H}$  the function

$$(4.9) \quad \alpha(t, \mathbf{p}, \mathbf{q}) = \int \exp \left\{ \frac{i}{\varepsilon} \mathbf{q} \cdot (\mathbf{p} - \mathbf{p}') \right\} \langle \mathbf{p} | \hat{A}(t) | \mathbf{p}' \rangle d\mathbf{p}'.$$

It is easy to see that the Heisenberg equations (3.7) lead to the following equation for the function (4.9):

$$(4.10) \quad i \frac{\partial \alpha(t, \mathbf{p}, \mathbf{q})}{\partial t} = \frac{1}{\varepsilon} \left( h \left( t, \mathbf{p} + i\varepsilon \frac{\partial}{\partial \mathbf{q}} \right) + h(t, \mathbf{p}) \right) \alpha(t, \mathbf{p}, \mathbf{q}).$$

In particular, for the vector function

$$(4.11) \quad \psi_{\mathbf{p}, \mathbf{q}}(t) = \int \exp \left\{ \frac{i}{\varepsilon} \mathbf{q} \cdot (\mathbf{p} - \mathbf{p}') \right\} \langle \mathbf{p} | \hat{a}(t) | \mathbf{p}' \rangle d\mathbf{p}'$$

Equation (4.10) can be written in the form

$$(4.12) \quad i \frac{\partial \psi_{\mathbf{p}, \mathbf{q}}(t)}{\partial t} = \frac{1}{\varepsilon} (h(t, \mathbf{p} + \varepsilon \mathbf{P}) - h(t, \mathbf{p})) \psi_{\mathbf{p}, \mathbf{q}}(t),$$

where we have noted that  $i \frac{\partial \psi_{\mathbf{p}, \mathbf{q}}(t)}{\partial \mathbf{q}} = P_j \psi_{\mathbf{p}, \mathbf{q}}(t)$  in accordance with the relation  $[\hat{a}^* P_j \hat{a}, \hat{a}] = P_j \hat{a}$ . The solution of the linear equation (4.12) with the initial condition  $\varphi_{\mathbf{p}} = \psi_{\mathbf{p}, 0}(t_0)$  can be written in the form (4.8), as we wish to prove. ■

If the function  $h(t, \mathbf{p})$  in the limit  $\varepsilon \rightarrow 0$  has a  $\mathbf{p}$ -differentiable limit, then (4.8) obviously takes the form of the soliton solution (2.14) of the Hartree equation. In this case, for  $\varepsilon \neq 0$ , the vector function (4.8) can be called an  $\varepsilon$  soliton, or an extremon [3].

## 5. APPENDIX

Taking into account equations (3.4) and (3.9), we write the representing functions

$$(5.1) \quad (\gamma \cdot \alpha)(\rho) = \exp \left\{ -\frac{1}{\varepsilon} \langle \rho, I \rangle \right\} \sum_{n=0}^{\infty} \frac{1}{n! \varepsilon^{n-1}} \left\langle \rho^{\otimes n}, H^{(n)} \cdot A^{(n)} \right\rangle,$$

$$(5.2) \quad \{\gamma, \alpha\}(\rho) = \exp \left\{ -\frac{1}{\varepsilon} \langle \rho, I \rangle \right\} \sum_{n=0}^{\infty} \frac{1}{n! \varepsilon^{n-1}} \left\langle \rho^{\otimes n}, \{H^{(n)}, A^{(n)}\} \right\rangle$$

for the products  $H^{(n)} \cdot A^{(n)}$ ,  $\{H^{(n)}, A^{(n)}\}$  in the symbolic form

$$e^{-\frac{1}{\varepsilon} \langle \rho, I \rangle} \sum_{n=0}^{\infty} \frac{1}{n! \varepsilon^{n+1}} \left\langle \rho^{\otimes n}, (I + \varepsilon \delta_1)^{\otimes n} \cdot (I + \varepsilon \delta_2)^{\otimes n} \right\rangle \gamma(\rho_1) \alpha(\rho_2) |_{\rho_1=0=\rho_2},$$

$$e^{-\frac{1}{\varepsilon}\langle\rho, I\rangle} \sum_{n=0}^{\infty} \frac{1}{n! \varepsilon^{n+1}} \langle \rho^{\otimes n}, \{(I + \varepsilon \delta_1)^{\otimes n}, (I + \varepsilon \delta_2)^{\otimes n}\} \rangle \gamma(\rho_1) \alpha(\rho_2) |_{\rho_1=0=\rho_2},$$

where  $\delta_1$  and  $\delta_2$  are the derivatives with respect to  $\rho_1$  and  $\rho_2$  respectively. Using further Equations. 3.2 for the powers  $H^{(n)} = (I + \varepsilon \delta_1)^{\otimes n}$ ,  $A^{(n)} = (I + \varepsilon \delta_2)^{\otimes n}$ , we obtain

$$(5.3) \quad \begin{aligned} \langle \rho^{\otimes n}, (I + \varepsilon \delta_1)^{\otimes n} \cdot (I + \varepsilon \delta_2)^{\otimes n} \rangle &= \\ \langle \rho, (I + \varepsilon \delta_1) \cdot (I + \varepsilon \delta_2) \rangle^n, \end{aligned}$$

$$(5.4) \quad \begin{aligned} \langle \rho^{\otimes n}, \{(I + \varepsilon \delta_1)^{\otimes n}, (I + \varepsilon \delta_2)^{\otimes n}\} \rangle &= \\ n \langle \rho, (I + \varepsilon \delta_1) \cdot (I + \varepsilon \delta_2) \rangle^{n-1} \langle \rho, \{\varepsilon \delta_1, \varepsilon \delta_2\} \rangle. \end{aligned}$$

Substituting this into the preceding formulas, we obtain

$$(\gamma \cdot \alpha)(\rho) = \frac{1}{\varepsilon} e^{\frac{1}{\varepsilon}\langle\rho, (I + \varepsilon \delta_1) \cdot (I + \varepsilon \delta_2) - I\rangle} \gamma(\rho_1) \alpha(\rho_2) |_{\rho_1=0=\rho_2},$$

$$\{\gamma, \alpha\}(\rho) = e^{\frac{1}{\varepsilon}\langle\rho, (I + \varepsilon \delta_1) \cdot (I + \varepsilon \delta_2) - I\rangle} \langle \rho, \{\delta_1 \gamma(\rho_1), \delta_2 \alpha(\rho_2)\} \rangle |_{\rho_1=0=\rho_2}.$$

Multiplying the exponentials and noting that  $e^{\langle\rho, \delta'\rangle} \beta(\rho') |_{\rho'=0} = \beta(\rho)$ , we obtain

$$(5.5) \quad (\gamma \cdot \alpha)(\rho) = \frac{1}{\varepsilon} e^{\varepsilon\langle\rho, \delta_1 \cdot \delta_2\rangle} \gamma(\rho_1) \alpha(\rho_2) |_{\rho_1=\rho=\rho_2}$$

$$(5.6) \quad \{\gamma, \alpha\}(\rho) = e^{\varepsilon\langle\rho, \delta_1 \cdot \delta_2\rangle} \langle \rho, \{\delta_1 \gamma(\rho_1), \delta_2 \alpha(\rho_2)\} \rangle |_{\rho_1=\rho=\rho_2}.$$

Expanding the exponential (5.5) in a series in  $\varepsilon$ , we obtain the expansion (3.13). The first formula (3.14), from which Proposition (1) follows, is obtained by applying the second formula (5.5) to (5.6), and the expansion (3.14) in powers of  $\varepsilon$  can be readily obtained by using formulas of the type (5.3) and (5.4) in reverse order in the exponential (5.6).

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